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A Decomposition Method for Solving the Nonlinear Klein–Gordon Equation

E. Y. DEEBA AND S. A. KHURI

Department of Computer and Mathematical Sciences, University of Houston-Downtown, Houston, Texas 77002

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In this paper, Adomian's decomposition scheme is presented as an alternate method for solving the nonlinear Klein–Gordon equation. The method is demonstrated by several examples. Comparing the scheme with existing collocation, finite difference and finite element techniques shows that the present approach is highly accurate and converges rapidly. © 1996 Academic Press, Inc.

1. INTRODUCTION

Recently, Adomian's decomposition scheme is emerging as an alternate method for solving a wide range of problems whose mathematical models yield equations or system of equations involving algebraic, differential, integral, integro-differential, or differential-delay terms (for example, see [1–16]). It has been shown that the method yields rapidly convergent series solutions to linear and nonlinear deterministic and stochastic equations. In some instances, if the problem is nonlinear, then the linearization of the perturbation method yields a set of partial differential equations that needs to be solved. On the other hand, the decomposition method provides a direct scheme for solving the underlying problem and does not require linearization. We shall, among other things, present this in the paper.

Our attention will focus on the nonlinear Klein–Gordon equation of the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) + bu(x,t) + g(u(x,t)) = f(x,t) \quad (1.1)$$

$$u(x,0) = a_0(x), \quad \frac{\partial u}{\partial t}(x,0) = a_1(x) \quad (1.2)$$

with

$$x = (x_1, x_2, ..., x_m) \in R^m, \quad t \in (0, T],$$

where

$$\Delta = \sum_{j=0}^{m} \frac{\partial^2}{\partial x_j^2} \quad (1.3)$$

and b is real, g is a given nonlinear function, and f is a known function.

Equations (1.1)–(1.2) is one of the important mathematical models in quantum mechanics [23–24] and it occurs in relativistic physics [20–21] as a model of dispersive phenomena. There are numerous papers dealing with the existence, uniqueness of the smooth and weak solutions of (1.1)–(1.2) (see [18]), and with the numerical solutions using finite difference, finite element, or collocation methods such as in [17, 19, 22].

2. DECOMPOSITION METHOD AND KLEIN–GORDON EQUATION

In this section we shall describe the main algorithm of Adomian's decomposition method as it applies to a general nonlinear equation of the form

$$u - N(u) = f, \quad (2.1)$$

where N is a nonlinear operator on a Hilbert space H, f is a known element of H, and we are seeking $u \in H$ satisfying (2.1). We assume that for every $f \in H$, Eq. (2.1) has a unique solution.

The decomposition technique consists of representing the solution of (2.1) as a series

$$u = \sum_{n=0}^{\infty} u_n \quad (2.2)$$

and the nonlinear operator N is decomposed as

$$N(u) = \sum_{n=0}^{\infty} A_n, \quad (2.3)$$

where the A_n's are Adomian's polynomials of u_0, ..., u_n given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, \ldots . \quad (2.4)$$

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Upon substituting Eq. (2.2) and (2.3) into the functional equation (2.1) yields

$$\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n = f. \tag{2.5}$$

The convergence of the series in (2.5) will yield

$$\begin{aligned} u_0 &= f \\ u_1 &= A_0 \\ u_2 &= A_1 \\ &\vdots \\ u_n &= A_{n-1} \\ &\vdots \end{aligned} \tag{2.6}$$

Thus, one can recurrently determine every term of the series $\sum_{n=0}^{\infty} u_n$. The convergence of this series has been established (see [13, 14]). In [14] a proof of convergence is established using fixed point theorems. In [13] the hypotheses for proving convergence are less restrictive and are generally satisfied in physical problems. The two hypotheses that are necessary for proving convergence of Adomian's technique as given in [13] are:

1. The nonlinear functional equation (2.1) has a series solution $\sum_{n=0}^{\infty} u_n$ such that $\sum_{n=0}^{\infty} (1+\varepsilon)^n |u_n| < \infty$, where $\varepsilon > 0$ may be very small.

2. The nonlinear operator $N(u)$ can be developed in series according to u: $N(u) = \sum_{n=0}^{\infty} \alpha_n u^n$.

To illustrate the scheme, let $N(u)$ be a nonlinear function of u, say $g(u)$, where

$$u = u_0 + \lambda u_1 + \lambda^2 u_2 + \ldots$$

then the first four Adomian's polynomials A_n are given by

$$\begin{aligned} A_0 &= g(u(\lambda))|_{\lambda=0} = g(u_0) \\ A_1 &= (dg/du)(du/d\lambda)|_{\lambda=0} \\ A_2 &= \tfrac{1}{2}[(d^2h/du^2)(du/d\lambda)^2 + (dg/du)(d^2u/d\lambda^2)]|_{\lambda=0} \\ A_3 &= \tfrac{1}{6}[(d^3g/du^3)(du/d\lambda)^3 + 2(d^2g/du^2)(du/d\lambda)(d^2u/d\lambda^2) \\ &\quad + (d^2g/du^2)(d^2u/d\lambda^2)(du/d\lambda) \\ &\quad + (dg/du)(d^3u/d\lambda^3)]|_{\lambda=0}. \end{aligned} \tag{2.7}$$

The A_n's can finally be written in the following convenient way

$$A_n = \sum_{\nu=1}^{n} c(\nu, n) g^{(\nu)}(u_0). \tag{2.8}$$

The result is

$$\begin{aligned} A_0 &= g(u_0) \\ A_1 &= u_1 \frac{d}{du_0} g(u_0) \\ A_2 &= u_2 \frac{d}{du_0} g(u_0) + \frac{u_1^2}{2!} \frac{d^2}{du_0^2} g(u_0) \\ A_3 &= u_3 \frac{d}{du_0} g(u_0) + u_1 u_2 \frac{d^2}{du_0^2} g(u_0) + \frac{u_1^3}{3!} \frac{d^3}{du_0^3} g(u_0). \end{aligned} \tag{2.9}$$

How do we interpret and solve the Klein–Gordon equation in this setting? Following the Adomian decomposition analysis [4-5], define the linear operators

$$L_t = \frac{\partial^2}{\partial t^2}, \quad L_{x_i} = \frac{\partial^2}{\partial x_i^2}, \quad i = 1, 2, ..., m. \tag{2.10}$$

Consequently, (1.1) can be rewritten in terms of the operators (2.10) as

$$L_t u = \sum_{i=1}^{m} L_{x_i} u - bu - g(u) + f. \tag{2.11}$$

It was shown in [18] that Eq. (1.1) with conditions (1.2) possesses a unique solution. Thus the inverse operator of L_t, namely L_t^{-1}, exists and is the twofold indefinite integral; i.e.,

$$[L_t^{-1} f](t) := \int_0^t du \int_0^u dv\, f(v). \tag{2.12}$$

Operating on both sides of (2.11) with L_t^{-1} yields

$$L_t^{-1} L_t u = \sum_{i=0}^{m} L_t^{-1} L_{x_i} u - b L_t^{-1} u - L_t^{-1}(g(u)) + L_t^{-1} f \tag{2.13}$$

from which it follows, upon using the initial conditions given in (1.2),

$$\begin{aligned} u(x,t) = a_0(x) + a_1(x)t + \sum_{i=0}^{m} L_t^{-1} L_{x_i} u \\ - b L_t^{-1} u - L_t^{-1}(g(u)) + L_t^{-1} f. \end{aligned} \tag{2.14}$$

The Adomian decomposition method yields the solution in the form given in (2.2). The first term u_0 can be determined as

$$u_0(x,t) = a_0(x) + a_1(x)t + L_t^{-1}(f(x,t)). \tag{2.15}$$

If we set $N(u) = g(u) = \sum_{n=0}^{\infty} A_n$, then the next iterates are determined via the recursive relation

$$u_{n+1} = \sum_{i=0}^{m} L_t^{-1}L_{x_i}u_n - bL_t^{-1}u_n - L_t^{-1}A_n, \quad n \geq 0. \tag{2.16}$$

Hence, using (2.9), the first three iterates are given by

$$\begin{aligned}
u_1 &= \textstyle\sum_{i=0}^{m} L_t^{-1}L_{x_i}u_0 - bL_t^{-1}u_0 - L_t^{-1}(g(u_0)) \\
u_2 &= \textstyle\sum_{i=0}^{m} L_t^{-1}L_{x_i}u_1 - bL_t^{-1}u_1 - L_t^{-1}(u_1 g(u_0)) \\
u_3 &= \textstyle\sum_{i=0}^{m} L_t^{-1}L_{x_i}u_2 - bL_t^{-1}u_2 \\
&\quad - L_t^{-1}\left(u_2 \frac{d}{du_0} g(u_0) + \frac{u_1^2}{2!}\frac{d^2}{du_0^2} g(u_0)\right).
\end{aligned} \tag{2.17}$$

For later numerical computation, let the expression

$$\phi_n = \sum_{i=0}^{n-1} u_i(x,t) \tag{2.18}$$

denote the n-term approximation to u. Since the series converges very rapidly, the sum $\phi_n = \sum_{i=0}^{n-1} u_i$ can serve as a practical solution.

We will show through several examples, that the number of terms required to obtain an accurate computable solution is very small.

3. ILLUSTRATION OF THE METHOD

In this section we shall consider four examples where in (1.1) g is assumed to be either u, u^2, u^3, or $\sin u$, respectively. The outcome of Adomian's decomposition method shall be compared with any known solution to the underlying Klein–Gordon equation. The solutions obtained are generated by using MAPLE.

EXAMPLE 1. Consider the Klein–Gordon equation of the form

$$\begin{aligned}
u_{tt} - u_{xx} - 2u &= -2\sin x \sin t \\
u(x,0) = 0, \quad u_t(x,0) &= \sin x.
\end{aligned} \tag{3.1}$$

The term 2 sin x sin t will be shown to be a **noise term**.

Equation (2.14) implies that

$$\begin{aligned}
u(x,t) = t\sin x + L_t^{-1}L_x u + 2L_t^{-1}u \\
+ L_t^{-1}(-2\sin x \sin t).
\end{aligned} \tag{3.2}$$

If $u(x,t) = \sum_{n=0}^{\infty} u_n$, then (2.15) and (2.17) imply that the various iterates can be determined as

$$\begin{aligned}
u_0 &= t\sin x + L_t^{-1}(-2\sin x \sin t) \\
&= 2\sin x \sin t - t\sin x
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
u_1 &= L_t^{-1}L_x u_0 + 2L_t^{-1}u_0 \\
&= L_t^{-1}(L_x(2\sin x \sin t - t\sin x)) \\
&\quad + 2L_t^{-1}(2\sin x \sin t - t\sin x) \\
&= L_t^{-1}(2\sin x \sin t - t\sin x) \\
&= -2\sin t \sin x + 2t\sin x - \frac{1}{3!}t^3 \sin x.
\end{aligned} \tag{3.4}$$

In a like manner, we find

$$\begin{aligned}
u_2 &= L_t^{-1}L_x u_1 + 2L_t^{-1}u_1 \\
&= 2\sin x \sin t - 2t\sin x + \tfrac{1}{3}t^3 \sin x - \frac{t^5}{5!}\sin x
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
u_3 &= L_t^{-1}L_x u_2 + 2L_t^{-1}u_2 \\
&= -2\sin x \sin t + 2t\sin x - \tfrac{1}{3}t^3 \sin x \\
&\quad + \frac{2}{5!}t^5 \sin x - \frac{t^7}{7!}\sin x.
\end{aligned} \tag{3.6}$$

Also,

$$\begin{aligned}
u_4 &= L_t^{-1}L_x u_3 + 2L_t^{-1}u_3 \\
&= 2\sin x \sin t - 2t\sin x + \tfrac{1}{3}t^3 \sin x \\
&\quad - \frac{2}{5!}t^5 \sin x + \frac{2}{7!}t^7 \sin x - \frac{1}{9!}t^9 \sin x.
\end{aligned} \tag{3.7}$$

Upon summing these iterates, we observe that

$$\phi_4 = \sum_{i=0}^{3} u_i = \sin x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right) \tag{3.8}$$

and

$$\phi_5 = \sum_{i=0}^{4} u_i = 2 \sin x \sin t - \sin x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!}\right). \tag{3.9}$$

This explains the phenomena that 2 sin x sin t is the self-canceling ‘‘noise’’ term. Further, canceling the noise term we obtain inductively the exact solution to (3.1) given by

$$u(x, t) = \sin x \sin t. \tag{3.10}$$

Table I shows the errors obtained upon solving the Klein–Gordon equation (3.1) using only three terms of the decomposition method. In Table I and thereafter, error is understood to be $=$ |*true value* $-$ *approximate value*|. Hence, in Table I error $= |\sin x \sin t - \phi_3|$. It is noted that only three terms were needed to obtain an error of less than 1%. The overall errors can be made much smaller by adding new terms of the decomposition.

Our next example (see [20, 24]) focuses on a nonlinear problem that cannot be solved explicitly. The standard method that is used to handle such an equation is the perturbation method which reduces the underlying equation to a class of linear partial differential equations. We will present the decomposition method as an alternative for solving the equation.

EXAMPLE 2. Consider the hyperbolic equation

$$w_{tt} - \gamma^2 w_{xx} + c^2 w - \sigma w^3 = 0, \tag{3.11}$$

where γ, c, σ are appropriate physical constants, with the initial conditions

$$w(x, 0) = \varepsilon \cos kx, \quad w_t(x, 0) = 0, \quad -\infty < x < \infty \tag{3.12}$$

TABLE I

Error Obtained Using Decomposition Method with Three Iterations

x	$t = 0.2$	$t = 0.4$	$t = 0.6$
0.2	2.6×10^{-4}	5.2×10^{-4}	7.5×10^{-4}
0.4	2.10×10^{-3}	4.12×10^{-3}	5.98×10^{-3}
0.6	7.03×10^{-3}	1.377×10^{-2}	1.9964×10^{-2}

with $0 < \varepsilon \ll 1$ and k is a specified constant. When $w = \varepsilon u$, the above equation reduces to

$$\begin{aligned} u_{tt} - \gamma^2 u_{xx} + c^2 u - \varepsilon^2 \sigma u^3 = 0 \\ u(x, 0) = \cos ks, \quad u_t(x, 0) = 0. \end{aligned} \tag{3.13}$$

Equation (3.13) is an example of (1.1) with g being a cubic function and $f(x, t) = 0$. Equation (2.14) implies that

$$\begin{aligned} u(x, t) = \cos ks + \gamma^2 L_t^{-1} L_x u \\ - c^2 L_t^{-1} u + \varepsilon^2 \sigma L_t^{-1}(u^3). \end{aligned} \tag{3.14}$$

For convenience later we set $\omega^2 = \gamma^2 k^2 + c^2$, which is the dispersion relation for the linear Klein–Gordon equation. The iterates $u(x, t) = \sum_{n=0}^{\infty} u_n$, upon using (2.15) and (2.17), are given by

$$u_0 = \cos kx \tag{3.15}$$

and

$$\begin{aligned} u_1 &= \gamma^2 L_t^{-1} L_x u_0 - c^2 L_t^{-1} u_0 + \varepsilon^2 \sigma L_t^{-1}(u_0^3) \\ &= -\frac{1}{2!} \omega^2 t^2 \cos kx + \varepsilon^2 \sigma t^2 (\tfrac{3}{8} \cos kx + \tfrac{1}{8} \cos 3kx) + O(\varepsilon^3). \end{aligned} \tag{3.16}$$

Similarly,

$$\begin{aligned} u_2 &= \gamma^2 L_t^{-1} L_x u_1 - c^2 L_t^{-1} u_1 + \varepsilon^2 \sigma L_t^{-1}(3u_0^2 u_1) \\ &= \frac{1}{4!} \omega^4 t^4 \cos kx - \frac{1}{4!} \varepsilon^2 \sigma t^4 [3\omega^2 \cos kx \\ &\quad + (3k^2\gamma^2 + c^2) \cos 3kx] + O(\varepsilon^3) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} u_3 &= \gamma^2 L_t^{-1} L_x u_2 - c^2 L_t^{-1} u_2 + \varepsilon^2 \sigma L_t^{-1}(3u_0^2 u_2 + 3u_0 u_1^2) \\ &= -\frac{1}{6!} \omega^6 t^6 \cos kx + \frac{\varepsilon^2 \sigma}{4.6!} t^6 [75\omega^4 \cos kx \\ &\quad + (129\gamma^4 k^4 + 90\gamma^2 c^2 k^2 + 25c^4) \cos 3kx] + O(\varepsilon 3). \end{aligned} \tag{3.18}$$

Finally,

$$\begin{aligned} u_4 &= \gamma^2 L_t^{-1} L_x u_3 - c^2 L_t^{-1} u_3 + \varepsilon^2 \sigma L_t^{-1}(3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3) \\ &= \frac{1}{8!} \omega^4 t^4 \cos kx - \frac{\varepsilon^2 \sigma}{2.7!} t^8 [39\omega^6 \cos kx \\ &\quad + (13c^6 + 93c^2\gamma^4 k^4 + 54c^4\gamma^2 k^2 \\ &\quad + 84\gamma^6 k^6) \cos 3kx] + O(\varepsilon^3). \end{aligned} \tag{3.19}$$

Combining these iterates yields

$$\left(1-\frac{1}{2!}\omega^2t^2+\frac{1}{4!}\omega^4t^4-\frac{1}{6!}\omega^6t^6+\frac{1}{8}\omega^8t^8\right)\cos kx$$
$$+\varepsilon^2\sigma\left[\frac{3}{8}t^2-\frac{3}{4!}\omega^2t^4+\frac{75}{4.6!}\omega^4t^6-\frac{39}{2.7!}\omega^6t^8\right]\cos kx$$
$$+\varepsilon^2\sigma\left[\frac{1}{8}t^2-\frac{1}{4!}(3k^2\gamma^2+c^2)t^4\right.$$
$$+\frac{1}{4.6!}(129\gamma^4k^4+90\gamma^2k^2c^2+25c^4)t^6 \tag{3.20}$$
$$=-\frac{1}{2.7!}(13c^6+93c^2\gamma^4k^4+54c^4\gamma^2k^2$$
$$\left.+84\gamma^6k^6)t^8\right]\cos 3kx+O(\varepsilon^3).$$

Inductively, the first term of the series is $\cos\omega t\cos kx$ while the other terms, upon comparing them with the results of the perturbation method, yield

$$u(x,t)=\cos\omega t\cos kx+\varepsilon^2\left[\frac{9\sigma}{32\omega}t\sin\omega t\right.$$
$$\left.+\frac{3\sigma}{128\omega^2}(\cos\omega t-\cos 3\omega t)\right]\cos kx$$
$$+\varepsilon^2\left[\frac{3\sigma}{128\gamma^2k^2}(\cos\omega t-\cos\lambda t)+\frac{\sigma}{128c^2}\right. \tag{3.21}$$
$$\left.(\cos\lambda t-\cos 3\omega t)\right]\cos 3kx+O(\varepsilon^3),$$

where

$$\lambda^2=9\gamma^2k^2+c^2.$$

Adomian's method led to the same approximate solution obtained by the perturbation method. While in the perturbation method, one needs to solve a class of linear partial differential equations and match the boundaries to obtain the solution; in the decomposition method, once the problem is properly set, one only needs to differentiate and integrate to obtain Adomian's polynomials A_n and consequently u.

Our third example involves the sine–Gordon equation.

EXAMPLE 3. Consider the pendulum-like equation

$$\frac{d^2u}{dt^2}=\sin u \tag{3.22}$$

with the following initial conditions:

$$u(0)=\pi,\quad \frac{du}{dt}(0)=-2. \tag{3.23}$$

Equation (3.22) arises from the general sine–Gordon equation

$$u_{tt}-u_{xx}-\sin u=0 \tag{3.24}$$

which is a model for the nonlinear meson fields with periodic properties for the unified description of mesons and their particle sources (see [21]). Simple solutions of (3.24) may be found in which u is a function of $\xi=(x-vt)/\sqrt{1-v^2}$. In this case, the field equation (3.24) reduces to the pendulum-like equation $d^2u/d\xi^2=\sin u$. For a real physical velocity $v<1$, there are implicit solutions given by $\sin\frac{1}{2}u=\pm\operatorname{sech}(\xi-\xi_0)$ which have particle number $N=\pm1$ and total energy $E=(2/\pi)(1/\sqrt{1-v^2})$. These may be interpreted as the fields associated with a particle of mass $2/\pi$ centered at $\xi=\xi_0$ and moving with velocity v [21].

We will show, using Adomian's decomposition method, how to obtain solutions that coincide with the implicit solutions of Eqs. (3.22)–(3.23) given by

$$\sin\tfrac{1}{2}u=\operatorname{sech} t. \tag{3.25}$$

We will first describe the decomposition scheme as it applies to the general sine–Gordon equation (3.24). If we set $L_t=\partial^2/\partial t^2$, $L_x=\partial^2/\partial x^2$, $N(u)=\sin u$, then Eq. (2.14) implies that the solution of (3.22) can be expressed in the operator form

$$u(x,t)-u(x,0)-tu_t(x,0)=L_t^{-1}L_xu+L_t^{-1}N(u). \tag{3.26}$$

Thus writing $u(x,t)=\sum_{n=0}^{\infty}u_n$, and $N(u)=\sin u=\sum_{n=0}^{\infty}A_n$ then, using (3.26) and (2.16), the various iterates are given by

$$u_{n+1}=L_t^{-1}L_xu_n+L_t^{-1}A_n,\quad n\geq 0 \tag{3.27}$$

with

$$u_0=u(x,0)+tu_t(x,0). \tag{3.28}$$

For $N(u) = f(u) = \sin u$ and upon using (2.9), we have

$$
\begin{aligned}
A_0 &= \sin u_0 \\
A_1 &= u_1 \cos u_0 \\
A_2 &= u_2 \cos u_0 - \frac{u_1^2}{2!} \sin u_0 \\
A_3 &= u_3 \cos u_0 - u_2 u_1 \sin u_0 - \frac{u_1^3}{3!} \cos u_0 \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned}
\tag{3.29}
$$

Therefore, since u_0 is known, Eqs. (3.27)–(3.28) provide the series solution $\sum_{n=0}^{\infty} u_n$, where

$$
\begin{aligned}
u_0 &= u(x,0) + t u_t(x,0) \\
u_1 &= L_t^{-1} L_x u_0 + L_t^{-1} A_0 \\
u_2 &= L_t^{-1} L_x u_1 + L_t^{-1} A_1 \\
&\cdot \\
&\cdot \\
u_{n+1} &= L_t^{-1} L_x u_n + L_t^{-1} A_n
\end{aligned}
\tag{3.30}
$$

with the A_n's given in (3.29).

We will adapt the above algorithm in (3.30) to the pendulum-like equation given in (3.22) with initial conditions (3.23), where u is only a function of t. For this special case, the polynomials A_n given in (3.29) are dependent on t only. The solution (3.27) thus reduces to

$$
u_{n+1} = L_t^{-1} A_n, \quad n \geq 0,
\tag{3.31}
$$

where

$$
u_0 = u(0) + t \frac{du}{dt}(0) = \pi - 2t.
\tag{3.32}
$$

Using (3.30) and (3.31)–(3.32) the various iterates are given as

$$
\begin{aligned}
u_1 &= L_t^{-1} A_0 = L_t^{-1}(\sin u_0) \\
&= \tfrac{1}{2} t - \tfrac{1}{4} \sin 2t.
\end{aligned}
\tag{3.33}
$$

In a like manner, we find upon using the identity $\cos(\pi - 2t) = -\cos 2t$,

$$
\begin{aligned}
u_2 &= L_t^{-1} A_1 = L_t^{-1}(u_1 \cos u_0) \\
&= -L_t^{-1}(u_1 \cos 2t) \\
&= \tfrac{5}{32} t - \tfrac{1}{8} \sin 2t + \tfrac{1}{8} t \cos 2t - \tfrac{1}{64} \sin 2t \cos 2t
\end{aligned}
\tag{3.34}
$$

and

$$
\begin{aligned}
u_3 &= L_t^{-1} A_2 = L_t^{-1}\left(u_2 \cos u_0 - \frac{u_1^2}{2!} \sin u_0 \right) \\
&= L_t^{-1}\{(-\cos 2t) u_2 + (-\tfrac{1}{2} \sin 2t) u_1^2\} \\
&= \tfrac{5}{64} t - \tfrac{47}{576} \sin 2t + \tfrac{13}{128} t \cos 2t \\
&\quad - \tfrac{1}{64} \sin 2t \cos 2t + \tfrac{1}{32} t^2 \sin 2t \\
&\quad + \tfrac{1}{64} t \cos^2 2t - \tfrac{1}{2304} \cos^2 2t \sin 2t + \tfrac{1}{1152} \sin^3 2t,
\end{aligned}
\tag{3.35}
$$

where in (3.35) the identities $\cos(\pi - 2t) = -\cos 2t$ and $\sin(\pi - 2t) = \sin 2t$ are used.

Higher iterates can be determined similarly. Upon combining the first six iterates and expanding in Taylor's series around $t = 0$, we obtain

$$
\begin{aligned}
u = \pi - 2t + \frac{2}{3!} t^3 - \frac{10}{5!} t^5 + \frac{61}{7!} t^7 \\
- \frac{2770}{9!} t^9 + \frac{103058}{11!} t^{11} + \ldots
\end{aligned}
\tag{3.36}
$$

which coincide with Taylor expansion of (3.25). We observed that each time an iterate is added, the Taylor expansion coincides to the next higher term.

Our final example deals with a nonhomogeneous Klein–Gordon equation having a quadratic nonlinear term.

EXAMPLE 4. Consider the Klein–Gordon equation of the form

$$
\begin{aligned}
u_{tt} - u_{xx} + \frac{\pi^2}{4} u + u^2 = x^2 \sin^2 \frac{\pi}{2} t \\
u(x,0) = 0, \quad u_t(x,0) = \frac{\pi}{2} x.
\end{aligned}
\tag{3.37}
$$

Equation (2.14) implies that

$$u(x,t) = \tfrac{1}{2}\pi xt + L_t^{-1}L_x u - \frac{\pi^2}{4}L_t^{-1}u - L_t^{-1}(u^2) + L_t^{-1}\left(x^2 \sin^2\frac{\pi}{2}t\right). \tag{3.38}$$

Thus if $u(x,t) = \sum_{n=0}^{\infty} u_n$, then using (2.15)–(2.17) the first three terms are given by

$$u_0 = \tfrac{1}{2}\pi xt + L_t^{-1}\left(x^2 \sin^2\frac{\pi}{2}t\right) = \tfrac{1}{2}\pi xt - \tfrac{1}{2}\frac{x^2}{\pi^2} + \tfrac{1}{4}\left(t^2 + \frac{2}{\pi^2}\cos \pi t\right)x^2 \tag{3.39}$$

and

$$u_1 = L_t^{-1}L_x u_0 - \frac{\pi^2}{4}L_t^{-1}u_0 - L_t^{-1}(u_0^2). \tag{3.40}$$

In a like manner,

$$u_2 = L_t^{-1}L_x u_1 - \frac{\pi^2}{4}L_t^{-1}u_1 - L_t^{-1}(2u_0u_1) \tag{3.41}$$

$$u_3 = L_t^{-1}L_x u_2 - \frac{\pi^2}{4}L_t^{-1}u_2 - L_t^{-1}(u_1^2 + 2u_0u_2). \tag{3.42}$$

The exact solution of (3.37) is given by

$$u(x,t) = x \sin\frac{\pi}{2}t. \tag{3.43}$$

The exact solution was compared with the approximate solution using Adomian's method. Tables II and III show the errors obtained by using the approximation ϕ_3 and ϕ_4, as defined in (2.18), respectively. It is evident that the error decreases as the number of iterations increase. Convergence is rapid and the errors are extremely small. The presence of self-canceling noise terms such as $-\frac{1}{2}(x^2/2\pi^2)$ and $\frac{1}{4}t^2x^2$ in Eq. (3.39) affects the monotonicity of the errors (see [3]).

TABLE II

Errors Obtained Using Decomposition Method with Three Iterations

t	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
0.1	1.2×10^{-11}	6.3×10^{-11}	1.2×10^{-10}	1.8×10^{-10}	2.4×10^{-10}
0.2	2.9×10^{-9}	4.2×10^{-9}	1.2×10^{-8}	2.1×10^{-8}	3.1×10^{-8}
0.3	1.3×10^{-7}	3.3×10^{-9}	1.6×10^{-7}	3.3×10^{-7}	5.3×10^{-7}
0.4	1.5×10^{-6}	4.9×10^{-7}	7.7×10^{-7}	2.3×10^{-6}	4.0×10^{-6}
0.5	9.9×10^{-6}	4.7×10^{-6}	1.8×10^{-6}	9.8×10^{-6}	1.9×10^{-5}

TABLE III

Errors Obtained Using Decomposition Method with Four Iterations

t	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
0.1	2.8×10^{-14}	7.0×10^{-15}	1.7×10^{-14}	4.6×10^{-14}	7.8×10^{-14}
0.2	3.7×10^{-11}	2.5×10^{-11}	8.6×10^{-12}	1.2×10^{-11}	3.7×10^{-11}
0.3	2.3×10^{-9}	1.8×10^{-9}	1.0×10^{-9}	4.2×10^{-11}	1.3×10^{-9}
0.4	4.3×10^{-8}	3.6×10^{-8}	2.5×10^{-8}	8.1×10^{-9}	1.4×10^{-8}
0.5	4.2×10^{-7}	3.6×10^{-7}	2.7×10^{-7}	1.2×10^{-7}	8.0×10^{-8}

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